J. Phys. A: Math. Gen. 28 (1995) 6277–6287. Printed in the UK

Finite-size scaling for random walks on fractals

Fábio D A Aarão Reis

Departamento de Física, Universidade Federal Fluminense, Av. Litorânea s/n, Gragoatá, CEP 24210-340, Niterói, RJ. Brazil

Received 28 June 1995

Abstract. Random walks are simulated on finite stages of construction of regular fractal lattices. It is proved that the mean-square displacement $\langle R_N^2 \rangle$ obeys a finite-size scaling hypothesis and the critical exponent ν_w is estimated. The efficiency of the method is proved when applied to finitely ramified fractals in which the problem is exactly solvable. ν_w is obtained with good accuracy ($\approx 1\%$) for a class of infinitely ramified fractals, the Sierpinski carpets. The results correct previous estimates based on simulations which did not use finite-size scaling. It is shown that ν_w decreases when D_F decreases with very small corrections due to other geometrical properties such as lacunarity. The comparison with estimates of the ideal chain exponent ν_c shows that the two problems are not equivalent on these fractals, and that in general $\nu_w > \nu_c$. Estimates of ν_w with the same accuracy are obtained on two Sierpinski pastry shells ($2 < D_F < 3$), where anomalous diffusion is also observed.

1. Introduction

Random walks on fractals have been studied intensively in the last few years especially due to their relation to diffusion in disordered systems [1]. Their most important property is the anomalous diffusion: the mean-square displacement of the walker varies with the number of steps N as

$$\langle R_N^2 \rangle \sim N^{2\nu_w} \tag{1}$$

with $\nu_w < \frac{1}{2}$ ($\nu_w = 1/D_w$, where D_w is the dimension of the random walk), while in Euclidean lattices $\nu_w = \frac{1}{2}$. The irregularities of the fractals are responsible for the delay of the diffusion [1].

In the random walk problem on a lattice, the walker at a certain site after $N-1$ steps has equal probability to jump to any of its neighbouring sites in the Nth step. The statistical weight of the walk depends on the sites it visits if the coordination number of the lattice is not uniform [1, 2].

The ideal chain problem is closely related to the random walk. It is defined on the same ensemble of walks, but the statistical weight of the chain depends only on its length (x^N, where x is the step fugacity). It is the equilibrium statistical problem of an ideal polymer (with no self-avoiding effects) in solution [3]. The mean square end-to-end distance of the ideal chain scales according to

$$\langle R_N^2 \rangle \sim N^{2\nu_c} . \tag{2}$$

In Euclidean lattices and also on fractals with uniform coordination number (e.g. the Sierpinski gasket), the random walk and the ideal chain have the same asymptotic behaviour, i.e. $\nu_w = \nu_c$ [1, 4].

0305-4470/95/226277+11$19.50

6. Conclusion

Finite-size scaling techniques were used to calculate the anomalous diffusion exponent ν_w in various regular fractals, using data from simulations on their finite stages of construction. Applications to fractals where the problem is exactly solvable proved that reliable and accurate estimates can be obtained.

For the Sierpinski carpets new results were presented. For example, the relation $\nu_w > \nu_c$ was found in many lattices, in contrast to the result $\nu_w < \nu_c$ in some finitely ramified fractals. It was also shown that the dependence of ν_w on D_F is much stronger than its dependence on other geometrical properties such as lacunarity, in contrast to some related systems (e.g. ideal chains or self-avoiding walks).

For Sierpinski pastry shells ($2 \leqslant D_F \leqslant 3$) estimates of $\nu_w < \frac{1}{2}$ were also obtained.

The comparison with other techniques to study critical phenomena on fractal lattices shows the advantage of finite-size scaling. Simulations on large lattices but not analysed with this technique give biased estimates of critical exponents and renormalization techniques for infinitely ramified fractals make approximations of the lattices. Series expansions methods, although considering the true fractal limit, generally do not provide such accurate results due to the small orders of the series. Also note the possibility of studying fractals with $D_F \approx 3$, which would be much more difficult with the other methods.

Other problems can be studied using this technique, such as self-avoiding walks on fractals. Then many classes of fractals can be studied and some open questions, the solutions of which depend on accurate estimates of critical parameters, may be answered. Work along these lines is in progress.

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